

# Operators Extending (Pseudo-)Metrics

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## Abstract

We introduce a general method of extending (pseudo-)metrics from  $X$  to  $FX$ , where  $F$  is a normal functor on the category of metrizable compacta. For many concrete instances of  $F$ , our method specializes to the known constructions.

## 1 Introduction

Consider the category of all compact metrizable spaces which will be referred to as  $MComp$ . All functors are expected to be *normal* (for the definition and properties see [2, page 165] or [3]) and to have  $MComp$  as both the domain and the codomain. For a normal functor  $F$ , every space  $X$  is naturally embeddable in  $FX$ , so further in this work  $X$  is considered to be a subspace of  $FX$ .

By an *operator*  $u : C(-) \rightarrow C(F(-))$  we mean a family of maps

$$(u_X : C(X) \rightarrow C(FX))_{X \in MComp},$$

where  $C(X)$  denotes the set of all continuous mappings from  $X$  to  $\mathbb{R}$ . Considering different topologies on this set, one can speak about *operators continuous in the pointwise topology*, *in the uniform topology*, etc. An operator is called a *functorial operator* if for every  $i : Y \rightarrow X$  the following identity holds:

$$u_Y \circ i_* = (F(i))_* \circ u_X. \quad (1)$$

Here, for  $i : Y \rightarrow X$ , the mapping  $i_* : C(X) \rightarrow C(Y)$  corresponds  $\phi$  to  $\phi \circ i$ .

For  $f, g \in C(X)$  we write  $f \geq g$  to denote the pointwise inequality:  $f(x) \geq g(x)$  for all  $x \in X$ . An operator  $u$  is an *extension operator* if  $u_X(\phi)|_X = \phi$ ;

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*monotonous* if  $\phi \geq \psi$  implies  $u_X(\phi) \geq u_X(\psi)$ ; *semiadditive* if  $u_X(\phi + \psi) \leq u_X(\phi) + u_X(\psi)$ ; *positive* if  $\phi \geq 0$  implies  $u_X(\phi) \geq 0$ , all  $X$  and  $\phi, \psi \in C(X)$ .

Here we investigate a general method for extending (pseudo-)metrics from a metrizable compact  $X$  to  $FX$ , where  $F$  is a normal functor. For many concrete instances of  $F$ , our method specializes to the known constructions.

## 2 Definition and properties of the new operator

Suppose that we have a normal functor  $F$  and an operator  $u : C(-) \rightarrow C(F(-))$ . For  $a, b \in FX$ ,  $\langle a, b \rangle$  denotes the set

$$\{c \in F(X \times X) \mid Fpr_1(c) = a, Fpr_2(c) = b\} = (F(pr_1), F(pr_2))^{-1}(a, b).$$

It is not empty since any normal functor is bicommutative. Also, we will use some other notation:

$$\Delta_X : X \rightarrow X \times X, \quad \Delta(x) = (x, x); \quad (2)$$

$$\nabla_X : X \times X \rightarrow X \times X, \quad \nabla_X(x, y) = (y, x) \quad (3)$$

If no confusion arises, we simply write  $\Delta$  or  $\nabla$ .

For any real-valued function  $p$  on  $X^2$ , we may define a function  $\tilde{p}$  on  $(FX)^2$  by the following formula:

$$\tilde{p}(a, b) = \inf\{u_{X \times X}(p)(c) \mid c \in \langle a, b \rangle\}, \quad a, b \in FX \quad (4)$$

The formula (4) gives the promised operator  $\tilde{\cdot}$ . Of course, to define it, one needs an operator  $u$  first, so it seems that do not gain much. But, for many functors  $F$ , there is usually a natural and obvious definition of  $u$ , while it is typically not clear how to define a (pseudo-)metric on  $FX$  should we have one on  $X$ .

**Lemma 1** *If  $u$  is an extension operator, then the function  $\tilde{p}$  extends  $p$ .*

*Proof.* The claim is obvious because, for any normal functor  $F$  and arbitrary  $a, b \in X$ , the set  $\langle a, b \rangle \subset F(X \times X)$  consists of one point. ■

**Lemma 2** *If  $u$  is a positive, monotonous, semiadditive functorial operator, then for any pseudometric  $p$  on  $X$  the function  $\tilde{p}$  is a pseudometric on  $FX$ .*

*Proof.* For any pair  $(X \supset Y)$  and for every  $\phi \in C(X)$  such that  $\phi|_Y = 0$ , we have  $u_X(\phi)|_{FY} = 0$ . This can be deduced from (1) by letting  $i$  be the identity map  $Y \rightarrow X$ . Here,

$$u_X(\phi)|_{FY} = (Fi)_*(u_X(\phi)) = u_Y(i_*(\phi)) = u_Y(0) = 0.$$

Now we can prove that, for any  $a \in FX$ , we have  $\tilde{p}(a, a) = 0$ . Since  $p|_{\Delta(X)} = 0$  we have  $u_{X \times X}(p)|_{F\Delta(X)} = 0$ , and

$$0 \leq \tilde{p}(a, a) \leq u_{X \times X}(p)(F\Delta(a)) = 0.$$

The function  $\tilde{p}$  is symmetric, as

$$F\nabla(\langle b, a \rangle) = \langle a, b \rangle, \quad \forall a, b \in FX$$

and

$$\begin{aligned} \tilde{p}(a, b) &= \inf(u_{X \times X}(p)(\langle a, b \rangle)) = \inf(u_{X \times X}(p)(F\nabla(\langle b, a \rangle))) \\ &= \inf(u_{X \times X}(\nabla_*(p))(\langle b, a \rangle)) = \inf(u_{X \times X}(p)(\langle b, a \rangle)) = \tilde{p}(b, a). \end{aligned}$$

In this chain of equalities we used the symmetry of  $p$  (i.e.  $\nabla_*(p) = p$ ), the functoriality of  $u$  (i.e.  $u_{X \times X}(\nabla_*(p))(x) = F\nabla_* \circ u_{X \times X}(p)(x) = u_{X \times X}(p)(F\nabla(x))$ ) and the identity

$$F\nabla_* \circ u_{X \times X} = u_{X \times X} \circ \nabla_* : C(X \times X) \rightarrow C(F(X \times X)). \quad (5)$$

Let  $a, b$  and  $c$  be arbitrary points in  $FX$ . Choose  $x_1 \in \langle a, b \rangle$  and  $x_2 \in \langle b, c \rangle$ , such that  $\tilde{d}(a, b) = u_{X \times X}(x_1)$  and  $\tilde{d}(b, c) = u_{X \times X}(x_2)$ .  $F$  is bicommutative so there exists  $y \in F(X^3)$  such that  $Fpr_{12}(y) = x_1$  and  $Fpr_{23}(y) = x_2$ . Let  $x_3 = Fpr_{13}(y) \in \langle a, c \rangle$ . Then

$$\begin{aligned} \tilde{d}(a, c) &\leq u_{X \times X}(d)(x_3) = u_{X \times X}(d)(Fpr_{13}(y)) \\ &= u_{X^3}(d \circ pr_{13})(y) \leq u_{X^3}(d \circ pr_{12} + d \circ pr_{23})(y) \\ &\leq u_{X \times X}(Fpr_{12}(y)) + u_{X \times X}(Fpr_{23}(y)) = \tilde{d}(a, b) + \tilde{d}(b, c). \end{aligned}$$

The lemma is proved. ■

**Lemma 3** *If  $u$  is continuous in the uniform topology, then so is the operator  $\tilde{\cdot}$ .*

*Proof.* For any  $a, b \in FX$ , we have

$$\begin{aligned} \|u_{X \times X}(d_1) - u_{X \times X}(d_2)\|_\infty &\geq u_{X \times X}(d_1)(x_2, y_2) - u_{X \times X}(d_2)(x_2, y_2) \\ &\geq \tilde{d}_1(a, b) - \tilde{d}_2(a, b) \geq u_{X \times X}(d_1)(x_1, y_1) - u_{X \times X}(d_2)(x_1, y_1) \\ &\geq -\|u_{X \times X}(d_1) - u_{X \times X}(d_2)\|_\infty, \end{aligned}$$

where  $\tilde{d}_i(a, b) = u_{X \times X}(d_i)(x_i, y_i)$ ,  $i = 1, 2$ . Hence

$$\|\tilde{d}_1 - \tilde{d}_2\|_\infty \leq \|u_{X \times X}(d_1) - u_{X \times X}(d_2)\|_\infty$$

and the operator  $\tilde{\cdot}$  is continuous in the uniform topology. ■

**Lemma 4** *If the mapping*

$$H_X = (Fpr_1, Fpr_2) : F(X \times X) \rightarrow FX \times FX$$

*is open for any  $X \in MComp$ , then  $\tilde{d} : FX \times FX \rightarrow \mathbb{R}$  is continuous.*

*Proof.* In fact,  $\langle a, b \rangle = H_X^{-1}(a, b)$ . Mapping  $H_X$  is both open and closed as  $\text{dom}(H_X), \text{codom}(H_X) \in MComp$ . So the mapping

$$H_X^{-1} : FX \times FX \rightarrow \exp(F(X \times X))$$

is continuous. Also, for any fixed  $f \in C(X)$ , the infimum map  $\inf_f : \exp(X) \rightarrow \mathbb{R}$ , defined by  $\inf_f(A) = \inf f(A)$ , is continuous.

Putting this all together we obtain the required. ■

The direct consequence of Lemmas 1–4 is the following.

**Theorem 5** *If  $u_X$  is a positive, monotone, semiadditive functorial operator extending functions from  $X$  to  $FX$ , then the operator  $\sim$  defined by formula (4) extends pseudometrics from  $X$  to  $FX$ . Moreover, if  $u$  is continuous in the uniform topology, then so is the operator  $\sim$ ; if  $H_X$  is an open mapping for all  $X \in MComp$ , then the pseudometric  $\tilde{d}$  is continuous for every continuous pseudometric  $d$ . ■*

A remarkable fact about the above defined operator  $\sim$  is that in many cases it coincides with the well-known constructions, as we are going to demonstrate now.

### 3 Case $F = \exp$

Let  $F = \exp$  (the functor of all closed subsets equipped with the Vietoris topology, see [2, page 139].) We define  $u : C(-) \rightarrow C(\exp(-))$  by the formula  $u_X(\phi)(A) = \sup(\phi(A))$ ,  $\phi \in C(X)$ ,  $A \in \exp(X)$ .

**Theorem 6** *For every metric  $d$  on  $X$ , we have  $\tilde{d} = d_H$  (Hausdorff metric).*

*Proof.* Let  $A, B \in \exp(X)$ ,

$$M = d_H(A, B) = \inf\{\epsilon > 0 \mid A_\epsilon \supset B, B_\epsilon \supset A\},$$

where, for example,  $A_\epsilon = \{x \in X \mid d(x, A) \leq \epsilon\}$ .

Then either there is  $b \in B$  with  $d(b, A) = M$  or there is  $a \in A$  with  $d(a, B) = M$ . Since  $\text{pr}_1(C) = A$  and  $\text{pr}_2(C) = B$  for every  $C \in \langle a, b \rangle$ , we have  $u_{X \times X}(d)(C) \geq M$ , which implies  $\tilde{d} \geq d_H$ .

On the other hand, define

$$C = \{(a, b) \in A \times B \mid d(a, B) = d(a, b) \text{ or } d(A, b) = d(a, b)\}.$$

It is easy to prove that  $C \in \langle a, b \rangle$  and  $u_{X \times X}(d)(C) = M$ . Thus, we obtain that  $\tilde{d} = d_H$ . ■

## 4 Case $F = (-)^n$ .

To define an operator  $u$  one has to assign a certain number, given a real-valued function  $\phi$  on  $X$  and a sequence  $x_1, \dots, x_n \in X$ . It may be done in many ways but the following definitions are most interesting:

$$\begin{aligned} u_{X \times X}(\phi)(x_1, \dots, x_n) &= \left( \sum_{i=1}^n \phi(x_i)^p \right)^{1/p}, \quad p \geq 1; \\ u_{X \times X}(\phi)(x_1, \dots, x_n) &= \max_i (\phi(x_i)). \end{aligned}$$

The easy verification shows that corresponding operators  $\tilde{\phantom{u}}$  have the following appearance:

$$\begin{aligned} \tilde{d}(x, y) &= \left( \sum_{i=1}^n d(x_i, y_i)^p \right)^{1/p}; \\ \tilde{d}(x, y) &= \max_i (d(x_i, y_i)). \end{aligned}$$

## 5 Case $F = P$

Let  $P$  denote the functor of probability measures, see [1]. The topology on the space  $PX$  can be defined by means of the metric

$$\bar{d}(\mu, \nu) = \inf \{ \eta(d) \mid \eta \in P(X \times X), Ppr_1(\eta) = \mu, Ppr_2(\eta) = \nu \}, \quad \mu, \nu \in PX$$

Letting  $u_X(\phi)(\mu) = \mu(\phi)$ ,  $\mu \in PX$ ,  $\phi \in C(X)$ , one can see that the definitions of  $\bar{d}$  and  $d$  coincide.

## 6 Case of the free (free abelian) group functor

On the contrary to our default assumptions, here we suppose that the functor  $G(-)$ , the free group functor, is defined on the category of metrizable compacta with selected point. (The selected point plays the role of the identity in  $GX$ .)

The topology on the space  $GX$  may be defined in different ways. Among them are the constructions of Swierczkowski and Graev. To find distance between “words”  $A, B \in GX$  one has to find all *proper representations*  $A = \prod_{i=1}^n (a_i)^{\epsilon_i}$  and  $B = \prod_{i=1}^m (b_i)^{\sigma_i}$ ,  $a_i, b_i \in X$ ,  $\epsilon_i, \sigma_i = \pm 1$ , that is, representations which have the same number of letters and degrees coinciding exactly:  $n = m$  and  $\epsilon_i = \sigma_i$  for  $1 \leq i \leq n$ . Then

$$d_1(A, B) = \inf \left( \sum_{i=1}^n d(a_i, b_i) \right),$$

where the infimum is taken for all proper representations. This is Graev’s construction. That of Swierczkowski (let us denote it by  $d_2$ ) is nearly the same except we calculate the sum only for all *different* pairs  $(a_i, b_i)$ . Obviously,  $d_1 \geq d_2$ .

It turned out that these metrics can also be represented in the form (4) for suitable  $u$ . Indeed, for  $\phi \in C(X)$  and for  $A = \prod_{i=1}^n (a_i)^{\epsilon_i} \in FX$  (written in the reduced form), let

$$u_X(\phi)(A) = \sum_i \phi(a_i),$$

but in the first case we take sum for all  $i = 1, \dots, n$  and in the second for all different  $a_i$ 's. The points of the set  $\langle A, B \rangle$  are in the bijective correspondence with the proper representations, which sends  $C = \prod_{i=1}^n (c_i)^{\epsilon_i}$  (in the reduced form) to the representations  $A = \prod_{i=1}^n pr_1(c_i)^{\epsilon_i}$  and  $B = \prod_{i=1}^n pr_2(c_i)^{\epsilon_i}$ . Since

$$u_{X \times X}(d)(C) = \sum_i d(pr_1(c), pr_2(c))$$

we get the claimed result.

The case  $F = A$  (the free abelian group functor) is analogous. The interested reader should be able to transfer easily all results by himself.

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## References

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